

A note on perturbative aspects of Leigh-Strassler deformed $\mathcal{N} = 4$ SYM theory

Kallingalthodi Madhu* and Suresh Govindarajan†

Department of Physics

Indian Institute of Technology Madras

Chennai 600036 INDIA

Abstract

We carry out a perturbative study of the Leigh-Strassler deformed $\mathcal{N} = 4$ SYM theory in order to verify that the trihedral $\Delta(27)$ symmetry holds in the quantum theory. We show that the $\Delta(27)$ symmetry is preserved to two loops (at finite N) by explicitly computing the superpotential. The perturbative superpotential is not holomorphic in the couplings due to finite contributions. However, there exist coupling constant redefinitions that restore holomorphy. Interestingly, the same redefinitions appear (in the work of Jack, Jones and North) if one requires the three-loop anomalous dimension to vanish in a theory where the one-loop anomalous dimension vanishes. However, the two field redefinitions seem to differ by a factor of two.

*E-mail: madhu@physics.iitm.ac.in

†E-mail: suresh@physics.iitm.ac.in

1 Introduction

The Leigh-Strassler deformations of the $\mathcal{N} = 4$ SYM theory [1] are a class of $\mathcal{N} = 1$ super conformal field theories that are particularly interesting in the context of AdS/CFT correspondence. Though the gravity dual for the most general deformation of $\mathcal{N} = 4$ theory is not yet known, a subclass of deformations known as β -deformation has been well studied [2–5]. There have been perturbative studies of the states of the super conformal algebra, especially the chiral primary states [6, 7]. The Leigh-Strassler (LS) theory is conformal on a subspace of the coupling space defined by the matter couplings Y_{IJK} and the gauge coupling g . The conformal properties can be understood by studying the β -functions for the chiral couplings and the NSVZ β -function for the gauge coupling. The superpotential of the theory is protected from renormalizations by holomorphy. Hence, for the chiral couplings the β -functions must be proportional to the anomalous dimensions of the chiral superfields.

$$\beta_{IJK} \equiv \beta(Y_{IJK}) \sim Y_{LJK} \gamma_I^L + Y_{ILK} \gamma_J^L + Y_{IJL} \gamma_K^L, \quad (1.1)$$

For LS theory, the gauge β -function given by NSVZ [8] reduces to

$$\beta^{NSVZ}(g) = -\frac{g^3}{32\pi^2} \left[\frac{2N\gamma_I^I}{(N^2 - 1)(1 - g^2 N(16\pi^2)^{-1})} \right]. \quad (1.2)$$

This is again proportional to the anomalous dimensions. From Eqn. (1.1), the anomalous dimension matrix seem to have nine components for a theory with three flavors like the LS theory. However as we shall see, the symmetry of the LS action constrains this matrix to be proportional to unit matrix, giving rise to a single condition $\gamma = 0$ which defines the subspace on which the theory is conformal.

Classically, the LS theory has a discrete non-abelian symmetry given by trihedral $\Delta(27)$ group [9, 10]. In ref. [7] it was shown that the chiral primaries can be classified as representations of this $\Delta(27)$ group. Thus it is important to know whether the $\Delta(27)$ group is a symmetry of the quantum theory. For this, one has to show that the quantum corrected superpotential and the Kähler potential preserves this symmetry. Another aspect which is quite interesting to understand is whether conformal invariance and holomorphy of the theory is preserved quantum mechanically. In the computation of anomalous dimension of

scalar composite operators we find that the contribution from the non F -terms cancel when we impose the condition for conformal invariance. This suggests that conformal invariance of the LS theory may be sufficient to ensure holomorphicity.

In the following section we explain the Leigh-Strassler theory and its symmetries and write the superpotential in a useful form. We discuss in section 3, the role of $\Delta(27)$ in preserving the conformal invariance of the theory by studying the anomalous dimension. In section 4, we check the conformal properties of the theory by computing anomalous dimension upto three-loop, following ref. [11] and point out the existence of coupling constant redefinitions that preserve conformal invariance of the theory. Section 5, explains computation of two-loop effective superpotential. The two-loop contribution is not holomorphic in coupling constant h , as is expected with a 1PI effective superpotential. We point out that it is strikingly similar to the three-loop contribution to the anomalous dimension giving rise to the possibility that same field redefinitions preserve conformal invariance and holomorphy of the LS theory. We also briefly describe the two-loop effective Kähler potential in section 6 and show that it preserves the $\Delta(27)$ symmetry of the LS theory. We conclude in section 7 with remarks about the results of the paper. We give the details of our computations in the various appendices.

2 LS deformed $\mathcal{N} = 4$ Yang-Mills theory

The Lagrangian density of the Leigh-Strassler theory in terms of $\mathcal{N} = 1$ superfields is

$$\begin{aligned} \mathcal{L} = & \int d^2\theta d^2\bar{\theta} \operatorname{Tr} \left(e^{-gV} \bar{\Phi}_i e^{gV} \Phi_i \right) + \left\{ \frac{1}{2g^2} \int d^2\theta \left[\operatorname{Tr} \left(\mathcal{W}^\alpha \mathcal{W}_\alpha \right) \right. \right. \\ & \left. \left. + i h \operatorname{Tr} \left(e^{i\pi\beta} \Phi_1 \Phi_2 \Phi_3 - e^{-i\pi\beta} \Phi_1 \Phi_3 \Phi_2 \right) + \frac{i h'}{3} \operatorname{Tr} \left(\Phi_1^3 + \Phi_2^3 + \Phi_3^3 \right) \right] + h.c. \right\} \end{aligned} \quad (2.1)$$

All fields transform in the adjoint of $SU(N)$ and we assume that $N > 2$. Let $q \equiv e^{i\pi\beta}$ and $\bar{q} \equiv e^{-i\pi\beta}$. When β is real, then q and \bar{q} are complex conjugates of each other. The imaginary part of β can always be absorbed by a redefinition of h . We have also set $\Theta = 0$.

The theory has the symmetry of the trihedral group, $\Delta(27) \sim ((\mathbb{Z}_3)_R \times \mathbb{Z}_3) \rtimes \mathcal{C}_3$, which is a discrete non-Abelian subgroup of $SU(3) \subset SU(4)$ [9]. This is obtained

from the β -deformed theory by further breaking down the $U(1)^3$ symmetry. The action of $\Delta(27)$ on the fields of the theory is as follows:

$$\begin{aligned} h &: \Phi_1 \longrightarrow \Phi_1, \Phi_2 \longrightarrow \omega \Phi_2, \Phi_3 \longrightarrow \omega^2 \Phi_3 \\ \tau &: \Phi_1 \longrightarrow \Phi_2 \longrightarrow \Phi_3 \longrightarrow \Phi_1 \end{aligned}$$

where h generates \mathbb{Z}_3 and τ generates \mathcal{C}_3 and ω is a non-trivial cube-root of unity. Further $(\mathbb{Z}_3)_R$ is a sub-group of $U(1)_R$ – we assign charge +1 to all fields (this is 3/2 times their R -charge).

We can rewrite the superpotential by combining the three chiral superfields into one superfield and use one meta-index $I, J, K, L \dots$ representing the $SU(N)$ adjoint index a, b, c, d, \dots as well as the index $i, j, k, l, \dots = 1, 2, 3$ which labels the three chiral superfields. The Leigh-Strassler superpotential (the trace below is in the fundamental representation of $SU(N)$)

$$W_{LS} = \frac{f}{6} \epsilon^{ijk} \text{Tr}_F(\Phi_i \Phi_j \Phi_k) + \frac{1}{6} c^{ijk} \text{Tr}_F(\Phi_i \Phi_j \Phi_k), \quad (2.2)$$

where the fully symmetric tensor c^{ijk} is given by

$$c^{ijk} = \begin{cases} c_0, & i \neq j \neq k \neq i, \\ c_1, & i = j = k, \\ 0, & \text{otherwise.} \end{cases} \quad (2.3)$$

One can prove that *only* the above choice for c^{ijk} leads to a superpotential that is invariant under the trihedral group $\Delta(27)$. In particular, couplings such as c^{112} vanish and $c_1 = c^{111} = c^{222} = c^{333}$. Thus, if $\Delta(27)$ is to remain of symmetry of the quantum theory, such couplings must *not* arise in the quantum theory [12].

In order to be able to compare with the usual representation of the LS superpotential, we give the relationship to the usual parameters h, q, h' :

$$f = h(q + \bar{q}) \quad , \quad c_0 = h(q - \bar{q}) \quad , \quad c_1 = 2h' \quad . \quad (2.4)$$

In terms of the meta-index, the LS superpotential can be written as follows (matching the notation of [11]):

$$W_{LS} = \frac{1}{6} Y^{IJK} \Phi_I \Phi_J \Phi_K, \quad (2.5)$$

where

$$Y^{IJK} \equiv Y^{(ia)(jb)(kc)} = \frac{1}{2} \left(i f \epsilon^{ijk} \otimes f_{abc} + 2 c^{ijk} \otimes d_{abc} \right).$$

The generators of $SU(N)$ in the fundamental representation have been taken to satisfy the identity (with the normalization $\text{Tr}_F(T_a T_b) = \delta_{ab}$)

$$\text{Tr}_F(T_a T_b T_c) \equiv \frac{1}{2} [i f_{abc} + 2 d_{abc}] . \quad (2.6)$$

f_{abc} are the structure constants of $SU(N)$ and d_{abc} is the totally symmetric tensor.

It is interesting to observe the quantum mechanical properties of the measure in the LS theory before we begin our discussion of perturbative properties. As shown in [13, 14], the NSVZ β -function [8] can be viewed as arising from the non-trivial transformation of the measure of the path integral under rescaling of the chiral and vector superfields. For theories with matter fields in three flavors in the adjoint adjoint representation, the β_{NSVZ} is proportional to the anomalous dimension γ of the chiral superfield. The requirement of vanishing of the γ -function defines the subspace of the space of couplings where the theory remains conformal. Particularly interesting is the question of how the measure changes under the $\Delta(27)$ action. The measure of $\mathcal{N} = 4$ SYM theory is invariant under $SU(4)_R$. As the spectrum of the LS theory is identical to that of $\mathcal{N} = 4$ SYM theory, it must also be invariant under the action of trihedral group $\Delta(27)$ which is after all a subgroup of $SU(4)_R$.

3 Conformal invariance of the LS theory

The trihedral symmetry group, $\Delta(27)$, can be seen as a finite sub-group of $SU(3) \subset SL(3, \mathbb{C})$. An arbitrary gauge-invariant cubic superpotential involving three chiral superfields (transforming in the adjoint of $SU(N)$), Φ^i , consists of eleven independent (complex) couplings. Linear redefinitions of the three fields form the group $SL(3, \mathbb{C})$ while $SU(3)$ is the sub-group of $SL(3, \mathbb{C})$ which preserves the (diagonal) kinetic energy which is encoded in the tree-level Kähler potential $\bar{\Phi}^i \Phi_i$. By means of linear redefinitions, it is possible to set eight of the eleven couplings that appear in the superpotential to zero and obtain the form given in Eqn. (2.2). The trihedral group $\Delta(27)$ emerges as the subgroup of $SL(3, \mathbb{C})$ that preserves that form. If the Kähler potential also retains its diagonal form, then $\Delta(27)$ is a symmetry of the theory.

We will now show that the trihedral symmetry and gauge-invariance implies that $\gamma_J^I \propto \delta_J^I$. Recall that the only gauge-invariant $SU(N)$ tensor is δ_a^b . Thus the gauge-invariance requires that the matrix of anomalous dimensions be proportional to δ_b^a . Thus, we write

$$\gamma_{jb}^{ia} \equiv \gamma_j^i \delta_b^a ,$$

where we have separated the flavor indices from the gauge indices.

Recall, that invariance under $\Delta(27)$ implies that couplings such as c^{112} vanish and requires $c^{111} = c^{222} = c^{333}$. For this to remain so we need $\beta(c^{112}) = 0$ and $\beta(c^{111}) = \beta(c^{222})$ to all orders in the quantum theory. Consider $\beta(c^{112})$ – it is given by (using $Y^{1a\ 1b\ 2c} \sim c^{112} d_{abc}$)

$$\beta(Y^{1a\ 1b\ 2c}) \sim d_{abc} \left(c^{11k} \gamma_k^2 + 2c^{1k2} \gamma_k^1 \right) . \quad (3.1)$$

The vanishing of the RHS in the background values of c^{ijk} given in Eqn. (2.3) needs $\gamma_1^2 = 0$ and $\gamma_3^1 = 0$. Similarly, one can show that all off-diagonal terms vanish by considering the β functions for all c^{iik} with $i \neq k$. We still need to show that the diagonal matrix is proportional to the identity matrix. For this we consider

$$\beta(Y^{1a\ 1b\ 1c}) - \beta(Y^{2a\ 2b\ 2c}) \sim d_{abc} (\gamma_1^1 c^{111} - \gamma_2^2 c^{222}) . \quad (3.2)$$

This vanishes only when $\gamma_1^1 = \gamma_2^2$. Similar considerations also require $\gamma_1^1 = \gamma_3^3$. This completes the proof that $\gamma_j^i \propto \delta_j^i$. We can thus write

$$\gamma_J^I \equiv \gamma \delta_J^I . \quad (3.3)$$

Thus, the vanishing of all the β -functions imposes only *one* condition, i.e.,

$$\gamma(g, h, \beta, h') = 0 ,$$

in the space of coupling constants in the LS theory. Below, we explicitly verify that the matrix of anomalous dimensions satisfies Eqn. (3.3) to three loops by specializing the results of Jack, Jones and North(JJN) to the LS theory [11].

4 Computing the anomalous dimension

We write the γ function (anomalous dimension) as

$$\gamma = \gamma^{(1)} + \gamma^{(2)} + \gamma^{(3)} + \dots \quad (4.1)$$

where the superscript denotes order of the loop contribution. The answers are given in the \overline{MS} -scheme.

One has the following general expressions for $\gamma^{(1)}$ and $\gamma^{(2)}$ [15–19]: We follow the notation of JJN except that our gauge coupling constant g is $\sqrt{2}$ times theirs [11].

$$(16\pi^2)\gamma^{(1)}_J = \frac{1}{2}Y^{IKL}Y_{JKL} - g^2C(R)_J^I \equiv P_J^I \quad (4.2)$$

$$(16\pi^2)^2\gamma^{(2)}_J = \left(Y^{IMK}Y_{JMN} - g^2C(R)_J^K\delta_N^I\right)P_K^N + g^4C(R)_J^IQ \quad (4.3)$$

where $Y_{IJK} = (Y^{IJK})^*$, $Q = T(R) - 3C(G)$. We define $C(G)\delta_b^a = f^{acd}f_{bcd} = 2N\delta_b^a$, $T(R)\delta_{ab} = (T_aT_b)$ and $C(R)_J^I = (T_aT_a)_J^I$. Here R refers to the reducible representation given by three copies of the adjoint representation. Specializing the the LS theory where $Q = 0$ and

$$\frac{1}{2}Y^{IKL}Y_{JKL} = \frac{1}{2}N\delta_J^I \left[|f|^2 + (|c_0|^2 + \frac{|c_1|^2}{2})\frac{N^2-4}{N^2}\right], \quad (4.4)$$

$$= 2N\delta_J^I \left[|h|^2 - |h|^2\frac{|q - \bar{q}|^2}{N} + |h'|^2\frac{N^2-4}{2N^2}\right] \equiv \hat{P} \delta_J^I \quad (4.5)$$

The one-loop γ function for the fields is given by JJN to be (using $C(R)_J^I = 2N\delta_J^I$)

$$16\pi^2\gamma^{(1)}_J = 16\pi^2\gamma^{(1)} \delta_J^I = (\hat{P} - 2g^2N)\delta_J^I. \quad (4.6)$$

The vanishing of the one-loop γ function is then

$$\gamma^{(1)} = 0 \implies \boxed{N \left[|h|^2 - |h|^2\frac{|q - \bar{q}|^2}{N} + |h'|^2\frac{N^2-4}{2N^2} \right] - g^2N = 0.} \quad (4.7)$$

In the $\mathcal{N} = 4$ limit, this expression simplifies to $g^2 = |h|^2$ and also matches the expression given by Penati et. al. [4]. The two-loop correction is given by

$$(16\pi^2)^2 \gamma^{(2)}_J = [-2\hat{P} - 2g^2N][\hat{P} - 2g^2N]\delta_J^I, \quad (4.8)$$

also vanishes in the sub-space where $\gamma^{(1)} = 0$. This is the well-known result that one-loop finite theories are two-loop finite as well.

The three-loop γ -function does *not* vanish in the \overline{MS} scheme. It was computed by JJN who also showed that there exists a renormalization scheme wherein the three-loop gamma function vanishes provided the one-loop contribution does. In the $\gamma^{(1)} = 0$ sub-space, Parkes computed the three-loop gamma function [20]

$$\begin{aligned}
(16\pi^2)^3 \gamma_P^{(3)J} &= \kappa \frac{g^6}{2^3} [12C(R)C(G)^2 - 2C(R)^2C(G) - 10C(R)^3 - 4C(R)\Delta(R)] \\
&\quad + \kappa \frac{g^4}{2^2} [4C(R)S_1 - C(G)S_1 + S_2 - 5S_3] - \kappa \frac{g^2}{2} Y^* S_1 Y + \kappa \frac{M_I^J}{4}
\end{aligned} \tag{4.9}$$

where $\kappa = 6\zeta(3)$ and

$$\begin{aligned}
S_{1I}^J &= Y_{IMN} C(R)_P^M Y^{JPN} = 4N \hat{P} \delta_I^J, \\
(Y^* S_1 Y)_I^J &= Y_{IMN} S_{1P}^M Y^{JPN} = 8N^2 \hat{P}^2 \delta_I^J, \\
S_{2I}^J &= Y_{IMN} C(R)_P^M C(R)_Q^N Y^{JPQ} = 8N^2 \hat{P} \delta_I^J, \\
S_{3I}^J &= Y_{IMN} (C(R)^2)_P^M Y^{JPN} = 8N^2 \hat{P} \delta_I^J, \\
\Delta(R) &= \sum_{\alpha} C(R_{\alpha}) T(R_{\alpha}) = 12N^2, \\
M_I^J &= Y_{K_1 K_2 K_3} Y_{L_1 L_2 L_3} Y_{I M_1 M_2} Y^{J K_3 L_3} Y^{K_1 L_1 M_1} Y^{K_2 L_2 M_2}.
\end{aligned}$$

Above, we have given the values taken by the various terms for the LS theory except for M_I^J which involves a complicated expression and is given later. Putting in these expressions, we find that all g -dependent terms vanish in the $\gamma^{(1)} = 0$ subspace leaving behind a simple expression:

$$(16\pi^2)^3 \gamma_P^{(3)J} = \frac{\kappa}{4} M_I^J, \tag{4.10}$$

This is indeed an interesting result – it implies that (in the $\gamma^{(1)} = 0$ subspace) the only diagram which contributes to $\gamma^{(3)}$ in the LS theory is the only *non-planar* diagram (see Figure 1) that first appears at three-loop. This diagram vanishes in $\mathcal{N} = 4$ SYM theory.

An explicit computation reveals that M_I^J is indeed proportional to the identity matrix (see Appendix B for more details)

$$\begin{aligned}
M_I^J &= \frac{3\zeta(3)}{2} \frac{4 - N^2}{N(N^2 - 1)} \left[\frac{1}{2} \left(18|c_0|^2 |c_1|^2 + 2c_0^3 (2\bar{c}_0^3 + \bar{c}_1^3) + c_1^3 (2\bar{c}_0^3 + \bar{c}_1^3) \right) \left(1 - \frac{10}{N^2} \right) \right. \\
&\quad \left. + \left(4\bar{f}^2 (4c_0^3 \bar{c}_0 + 2c_1^3 \bar{c}_0 - 6c_0^2 |c_1|^2) + 4f^2 (4\bar{c}_0^3 c_0 + 2\bar{c}_1^3 c_0 - 6\bar{c}_0^2 |c_1|^2) \right) \right] \delta_I^J.
\end{aligned}$$

The above term clearly vanishes in the $\mathcal{N} = 4$ limit and also vanishes in the large- N limit reflecting the non-planar nature of the diagram.

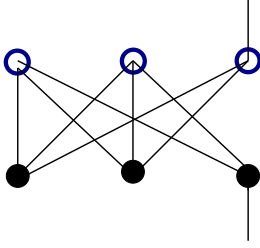


Figure 1: Non-planar contribution to $\gamma^{(3)}$ – a filled circle represents the chiral cubic vertex and a open circle represents an anti-chiral vertex.

4.1 Coupling constant redefinitions

In ref. [11], Jack, Jones and North have an interesting observation. They show that there exists a redefinition of the coupling constants for which the three-loop γ function also vanishes in a theory where $\gamma^{(1)} = 0$. This is equivalent to moving away from the \overline{MS} scheme. For the LS theory, due to the additional cancellations that we observed, the redefinition is simpler than the one used by JJN. One needs

$$(16\pi^2)^2 \delta Y_{IJK} = \frac{\kappa}{4} \mathcal{M}_{IJK} \quad (4.11)$$

where

$$\mathcal{M}_{KLM} = Y^{I_1 I_2 I_3} Y^{J_1 J_2 J_3} Y_{I_1 J_1 K} Y_{I_2 J_2 L} Y_{I_3 J_3 L} \quad (4.12)$$

On carrying out the coupling constant redefinition, the condition for conformal invariance continues to be the one given in Eqn. (4.7) albeit in the redefined couplings.

5 Two-loop effective superpotential

We next move on to the computation of the effective superpotential to two-loops. It was shown by West that in theories with massless fields such as the cubic Wess-Zumino model, that the 1PI superpotential is non-holomorphic in coupling constants due to finite contributions [21]. Such contributions do arise in our theory as well. We work out the coupling constant redefinition that is required to restore holomorphy. It turns out to be identical in structure to the one given in Eqn. (4.11) but is twice as large.

Below we give all the diagrams which can potentially contribute to the superpotential at two-loops. Diagrams (a)-(d) contribute terms that are proportional to the tree-level superpotential while (e) vanishes. All these diagrams also contribute to the $\mathcal{N} = 4$ theory. Diagram (f) is non-planar and leads to a non-holomorphic contribution to the superpotential. All the diagrams above lead to

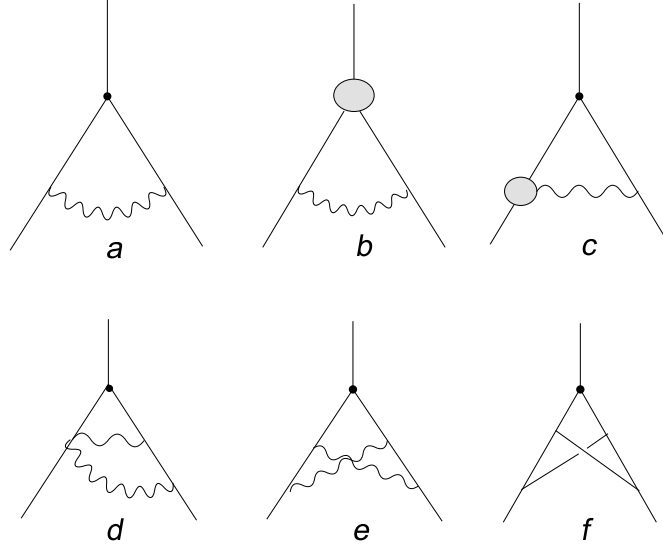


Figure 2: Contributions to the two-loop effective action. The blob that appear in (b) and (c) are one-loop vertex corrections.

finite integrals. For details of evaluation of these diagrams we refer to appendix C.

5.1 The non-planar diagram

The effective superpotential thus obtains a non-trivial contribution only from the diagram

$$\begin{aligned}
& \frac{1}{6^5} \times \frac{(3!)^5}{3!2!} \times \mathcal{M}^{IJK} \int d^2\theta_1 d^2\theta_2 d^2\theta_3 d^2\bar{\theta}_4 d^2\bar{\theta}_5 \\
& \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(p_2 + p_3, \theta_1) \Phi_J(-p_2, \theta_2) \Phi_K(-p_3, \theta_3)}{k^2 q^2 (k - q)^2 (q - p_2)^2 (k - q - p_2)^2 (k - p_2 - p_3)^2} \\
& \bar{D}_1^2 D_4^2[k] \delta^4(\theta_{14}) \bar{D}_2^2 D_4^2[q] \delta^4(\theta_{24}) \bar{D}_3^2 D_4^2[k - q] \delta^4(\theta_{34}) \\
& \bar{D}_1^2 D_5^2[k - p_2 - p_3] \delta^4(\theta_{15}) \bar{D}_2^2 D_5^2[q - p_2] \delta^4(\theta_{25}) \bar{D}_3^2 D_5^2[k - q - p_3] \delta^4(\theta_{35}) ,
\end{aligned} \tag{5.1}$$

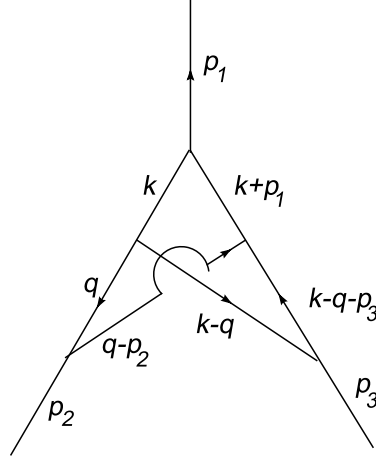


Figure 3: Chiral contribution to superpotential

where \mathcal{M}^{IJK} has been defined in Eqn. (4.12). Note that the momentum in the square brackets in the last two lines indicate the momentum appearing in the superderivatives. Details like the algebra of D -operators and simplification of the flavor and color factors in this computation are provided in the appendices B and C. We obtain the two-loop correction to the superpotential as

$$\mathcal{K} \frac{M^{IJK}}{12} \int d^2\theta \Phi_I(p_3, \theta) \Phi_J(p_3, \theta) \Phi_K(-p_3, \theta) \quad (5.2)$$

where \mathcal{K} is the finite integral

$$\mathcal{K} \equiv p_3^2 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{1}{k^2 r^2 (k-r)^2 (r-p_3)^2 (k-p_3)^2} = \frac{\kappa}{(16\pi^2)^2} \quad (5.3)$$

with $\kappa = 6\zeta(3)$. Putting in the explicit form of \mathcal{M}^{IJK} for the LS superpotential we obtain

$$\begin{aligned} \delta c_1 &= \mathcal{K} \left[\frac{-N^2+10}{2N^2} \right] \left[6|c_0|^4 c_1 + \bar{c}_1^2 (2c_0^3 + c_1^3) \right] - \mathcal{K} \left[6\bar{f}^2 c_0^2 c_1 \right] + 3\mathcal{K} \left[2f^2 (\bar{c}_1^2 c_0 - \bar{c}_0^2 c_1) \right] \\ \delta c_0 &= \mathcal{K} \left[\frac{-N^2+10}{2N^2} \right] \left[\bar{c}_0 (6|c_1|^2 c_0^2 + \bar{c}_0 (2c_0^3 + c_1^3)) \right] + \mathcal{K} \bar{f}^2 (2c_0^3 + c_1^3) + 6\mathcal{K} \left[f^2 \bar{c}_0 (|c_0|^2 - |c_1|^2) \right] \\ \delta f &= \mathcal{K} \left[\frac{-N^2+4}{2N^2} \right] \left[\bar{f} (-3|c_1|^2 c_0^2 + \bar{c}_0 (2c_0^3 + c_1^3)) \right] \end{aligned} \quad (5.4)$$

Specialising the above result to the β -deformed theory, it simplifies to the one given in the two-loop computation in ref. [6] (except for a mismatch of a factor of two).

5.1.1 Coupling constant redefinitions in the two-loop superpotential

The two-loop contribution to the effective superpotential thus leads to a redefinition of the form

$$(16\pi^2)^2 \delta Y^{IJK} = \frac{\kappa}{2} \mathcal{M}^{IJK} . \quad (5.5)$$

Holomorphy in the couplings is restored if we make a redefinition of the Y^{IJK} to absorb the non-holomorphic pieces in \mathcal{M}^{IJK} . We can compare this redefinition with the one required to make the gamma function vanish to three-loops given in Eqn. (4.11). It is interesting to note that both are proportional to $\kappa \mathcal{M}^{IJK}$ but *differ* by a factor of two. The result of [6] however requires the same redefinition – we have however been unable to find an error, if any, in our computation.

One may wish to know whether it is truly essential for the two redefinitions to agree. In principle, there is no such requirement. We could insist on holomorphy in couplings and choose the redefinition that is required by it. As the redefinition for conformal invariance is different, it implies that the condition of conformal invariance obtains a correction at three-loop and finite- N . So if one wishes to preserve the one-loop conformal invariance condition, then one needs to give up holomorphy in the couplings.

6 One-loop effective Kähler potential

The Kähler potential for any $\mathcal{N} = 1$ supersymmetric theory is non-holomorphic and provides the kinetic terms as well as the interactions between vector superfields with the chiral superfields. At tree-level in the LS theory, we have chosen the Kähler potential $\Phi_i \bar{\Phi}^i$. The effective one-loop Kähler potential has been computed in [22, 23] and we make use of their results – our notation is adapted from the second reference. In the Feynman gauge, the one-loop Kähler potential is given by

$$K_{\text{eff}}^{1\text{-loop}} = \sum_{n=1}^{\infty} \int \frac{d^4 k d^4 \theta}{(2\pi)^4} \frac{(-1)^{n+1}}{2n k^{2n+2}} \text{Tr} \left([\bar{\mu} \mu]^n - 2M^n \right) \quad (6.1)$$

where the first contribution arises from the insertion of n chiral and anti-chiral vertices and the second contribution arises from the insertion of n interaction

vertices involving the gauge field and the scalars. We have defined

$$\mu^{IJ} = Y^{IJK} \boldsymbol{\Phi}_K \quad , \quad \bar{\mu}_{IJ} = Y_{IJK} \bar{\boldsymbol{\Phi}}^K \quad , \quad M_{ab} = \frac{g^2}{2} \bar{\boldsymbol{\Phi}}^l \{T_a, T_b\} \boldsymbol{\Phi}_l \quad , \quad (6.2)$$

with the boldface $\boldsymbol{\Phi}$ indicating that the computation is being carried out in the background given by $\boldsymbol{\Phi}$.

The first term in Eqn. (6.1) is logarithmically divergent in the UV and is proportional to

$$\frac{1}{2} \text{Tr}(\bar{\mu}\mu) - \text{Tr}(M) = (16\pi^2)\gamma^{(1)} \bar{\boldsymbol{\Phi}}^L \boldsymbol{\Phi}_L \quad , \quad (6.3)$$

which vanishes in the conformal limit. This implies that there is *no* UV divergence in the integrals appearing in Eqn. (6.1). The appearance of the one-loop γ function in the $n = 1$ term is also not surprising since this is the term associated with the one-loop wavefunction renormalization. This will be true at higher orders as well. The trihedral symmetry also predicts that the quadratic correction to the Kähler potential will always be proportional $\bar{\boldsymbol{\Phi}}^L \boldsymbol{\Phi}_L$ due to the diagonal nature of the wavefunction renormalization.

The terms with $n > 1$ in Eqn. (6.1) are UV finite but are IR divergent. These are clearly suppressed by suitable powers of the UV cutoff and disappear in the conformal limit. The trihedral symmetry also imposes (less stringent) restrictions on the terms that can appear in these terms. We do not pursue this here.

7 Concluding Remarks

In this paper, we have shown that the trihedral group continues to remain a symmetry to two-loops in the quantum theory. We conjecture that it is a true symmetry of the LS deformed $\mathcal{N} = 4$ SYM theory. We also find an interesting relationship between holomorphy at two-loop and conformal invariance at three-loop – this appears due to the similarity in the coupling constant redefinitions. Ideally, one would like to think that the two are indeed the same. But the mismatch of a factor of two that we obtain seems to indicate a potential conflict. This mismatch can go away in two different ways – the three-loop anomalous dimension computation may be off by a factor of two or the two-loop superpotential may be incorrect. Given that the diagrams in question do not involve any gauge fields,

these issues can be addressed in the context of Wess-Zumino model. We carried out a detailed investigation of the literature in this context and interestingly discovered, in the context of the anomalous dimension, two different sets of results. Our conclusion is that the results of Jack, Jones and North (derived from the result of Parkes) is indeed correct. This leaves open the possibility that there is may be a factor of two error in our two-loop superpotential. We have been unable to find such an error and thus leave this issue for the future.

Acknowledgements We thank the anonymous referee of an earlier paper whose remarks lead us to the investigate the quantum aspects of the trihedral symmetry. We would also like to thank Justing David and in particular, Profs. Ian Jack and Tim Jones for extensive email correspondence regarding their work. The work of KM is supported by a Senior Research Fellowship from the CSIR (Award No. 9/84(327)/2001-EMR-I).

A Notations and conventions

We follow the notation of [24] through out this paper. The Greek indices $\mu, \nu \dots = 0, 1, 2, 3$ denote the space-time components and $\alpha, \beta = 1, 2$ and $\dot{\alpha}, \dot{\beta} = 1, 2$ are the $SU(2)$ spinor indices. The $i, j, k, \dots = 1, 2, 3$ run over the $SU(3)$ flavor indices and $a, b, c, \dots = 1, \dots, (N^2 - 1)$ are the $SU(N)$ color indices. The indices I, J, K, \dots is a combined notation for the flavor and color combination (i,a). The Minkowski metric is $g_{\mu\nu} = \text{diag}(+, -, -, -)$. Through out, we use the Weyl representation for the spinors. The undotted and dotted indices represent chiral and anti-chiral spinors. Spinors are raised or lowered as $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$, $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, $\psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}}$, $\psi_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\beta}}$, $\alpha = 1, 2$. Here $\epsilon_{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ are totally anti-symmetric tensors. The spinor summation convention is

$$\psi\chi = \psi^\alpha\chi_\alpha; \quad \bar{\psi}\bar{\chi} = \bar{\psi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} \quad (\text{A.1})$$

The square of a spinor is

$$\psi^2 = \frac{1}{2}\psi^\alpha\psi_\alpha; \quad \bar{\psi}^2 = \frac{1}{2}\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} \quad (\text{A.2})$$

The derivative with respect to the Grassmann coordinate is defined as

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}; \quad \partial_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad (\text{A.3})$$

The sigma matrices are

$$\begin{aligned}\sigma_0 = \bar{\sigma}^0 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & \sigma_1 = -\bar{\sigma}^1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 = -\bar{\sigma}^2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 = -\bar{\sigma}^3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}\quad (\text{A.4})$$

The superspace derivatives are

$$D_\alpha = \partial_\alpha + \frac{i}{2}\sigma_{\alpha\dot{\alpha}}^\mu \bar{\theta}^{\dot{\alpha}} \partial_\mu; \quad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} + \frac{i}{2}\theta^\alpha \sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu \quad (\text{A.5})$$

obeying the anti-commutation relation

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = i\sigma_{\alpha\dot{\alpha}}^\mu \partial_\mu. \quad (\text{A.6})$$

Further, $D^2 = -\frac{1}{2}D^\alpha D_\alpha$ and $\bar{D}^2 = -\frac{1}{2}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}$.

The integral over the Grassmann coordinates are defined such that

$$\int d^2\theta \, \theta^2 = \int d\theta^2 d\theta^1 \, \theta^1 \theta^2 = 1 = \int d^2\bar{\theta} \, \bar{\theta}^2 \quad (\text{A.7})$$

$$\bar{D}_1^2 D_1^2 [q] \delta^4(\theta_{12}) \Big|_{\theta_1=\theta_2} = 1 \quad (\text{A.8})$$

$$D_1^2 \bar{D}_1^2 \bar{D}_1^2 D_1^2 [q] \delta^4(\theta_{12}) \Big|_{\theta_1=\theta_2} = q^2 \quad (\text{A.9})$$

B Trace formulae for SU(N)

Below, we provide the trace identities and normalisations that we have used in our paper.

$$\begin{aligned}T^a T^a &= \frac{N^2 - 1}{N} I & \text{Tr}(T^a T^b) &= \delta^{ab} \\ \text{Tr}(AT^a BT^a) &= \text{Tr}(A)\text{Tr}(B) - \frac{1}{N}\text{Tr}(AB) \\ \text{Tr}(AT^a)\text{Tr}(BT^a) &= \text{Tr}(AB) - \frac{1}{N}\text{Tr}(A)\text{Tr}(B)\end{aligned}\quad (\text{B.1})$$

The following identities are useful in computing the one-loop anomalous dimension.

$$\epsilon^{ikl} \epsilon_{jkl} = 2\delta_j^i \quad (\text{B.2})$$

$$\bar{c}^{ikl} c_{jkl} = (2|c_0|^2 + |c_1|^2) \delta_j^i \quad (\text{B.3})$$

$$f^{acd} f_{bcd} = 2N \delta_b^a \quad (\text{B.4})$$

$$d^{acd} d_{bcd} = \left(\frac{N^2 - 4}{2N} \right) \delta_b^a \quad (\text{B.5})$$

The following identities involving five d/f tensors are required in the evaluation of \mathcal{M}^{IJK} :

$$\begin{aligned} d^{a_1 a_2 a_3} d^{b_1 b_2 b_3} d_{c_1 a_1 b_1} d_{c_2 a_2 b_2} d_{c_3 a_3 b_3} &= -\frac{N^2 - 10}{N^2} d_{c_1 c_2 c_3} \\ i f^{a_1 a_2 a_3} d^{b_1 b_2 b_3} d_{c_1 a_1 b_1} d_{c_2 a_2 b_2} d_{c_3 a_3 b_3} &= -\frac{N^2 - 4}{2N^2} i f_{c_1 c_2 c_3} \\ (i)^2 f^{a_1 a_2 a_3} f^{b_1 b_2 b_3} d_{c_1 a_1 b_1} d_{c_2 a_2 b_2} d_{c_3 a_3 b_3} &= 2 d_{c_1 c_2 c_3} \\ (i)^2 d^{a_1 a_2 a_3} d^{b_1 b_2 b_3} f_{c_1 a_1 b_1} f_{c_2 a_2 b_2} d_{c_3 a_3 b_3} &= 2 d_{c_1 c_2 c_3} \end{aligned} \quad (\text{B.6})$$

All other combinations involving five d/f tensors are *vanishing*.

Deriving the identities

We now sketch the method that we used to derive the various identities given in Eqn. (B.6). In the following, we represent $\text{Tr}(T_a T_b T_c)$ by (abc) . Further, we define

$$(\overline{abc}) = \frac{1}{2} [(abc) + (acb)] \quad , \quad (\widetilde{abc}) = \frac{1}{2} [(abc) - (acb)] \quad (\text{B.7})$$

Thus one has $d_{abc} = (\overline{abc})$ and $f_{abc} = \frac{2}{i} (\widetilde{abc})$. Let

$$[00000]_{klm} \equiv (a_1 a_2 a_3) (b_1 b_2 b_3) (k a_1 b_1) (l a_2 b_2) (m a_3 b_3).$$

We represent the $32 = 2^5$ combinations that can appear by a five bit number $[c_1 c_2 c_3 c_4 c_5]$ with the above equation defining $[00000]$. Each of the bits represents the five terms that appears in the RHS of the above equation. For instance, $c_1 = 0$ represents $(a_1 a_2 a_3)$ and $c_1 = 1$ represents $(a_1 a_3 a_2)$ and so on. There are symmetries which enables us to reduce the computation to only four independent terms which we then compute. The symmetries are as follows

1. $[c_1 c_2 c_3 c_4 c_5]_{klm} = [c_1 c_2 c_5 c_3 c_4]_{mkl} = [c_1 c_2 c_4 c_5 c_3]_{lmk}$.
2. $[c_1 c_2 c_3 c_4 c_5]_{klm} = [c_2 c_1 c_3 \oplus 1 c_4 \oplus 1 c_5 \oplus 1]_{klm}$ where $c_1 \oplus 1$ refers to the Boolean operation *xor*.
3. $[c_1 c_2 c_3 c_4 c_5]_{klm} = [c_1 \oplus 1 c_2 \oplus 1 c_3 c_4 c_5]_{kml}$.

Further isotropy of $[c_1 c_2 c_3 c_4 c_5]_{klm}$ under $SU(N)$ gauge transformations implies that

$$[c_1 c_2 c_3 c_4 c_5]_{klm} = A[c_1 c_2 c_3 c_4 c_5] (\overline{klm}) + B[c_1 c_2 c_3 c_4 c_5] (\widetilde{klm}) ,$$

where $A[c_1 c_2 c_3 c_4 c_5]$ and $B[c_1 c_2 c_3 c_4 c_5]$ are constants. The symmetries imply that we need to work out only four terms: $[00000]$, $[10001]$, $[10000]$, and $[10001]$. Using the identities given in Eqn. (B.1), we obtain

$$\begin{aligned} [00000]_{klm} &= \left[1 + \frac{10}{N^2}\right] (\overline{klm}) + \left[-1 + \frac{4}{N^2}\right] (\widetilde{klm}) \\ [00001]_{klm} &= \left[-1 + \frac{10}{N^2}\right] (\overline{klm}) + \left[-1 + \frac{4}{N^2}\right] (\widetilde{klm}) \\ [10000]_{klm} &= \frac{10}{N^2} (\overline{klm}) \\ [10001]_{klm} &= \left[-2 + \frac{10}{N^2}\right] (\overline{klm}) \end{aligned}$$

Using the above four relations we can work out all the 32 combinations. We can derive identities involving five combinations of the d and f $SU(N)$ tensors with this information. For instance, one has in order to obtain the identity involving five d tensors, we need to compute

$$\frac{1}{32} \sum_{c_1, \dots, c_5} A[c_1 c_2 c_3 c_4 c_5] \quad \text{and} \quad \frac{1}{32} \sum_{c_1, \dots, c_5} B[c_1 c_2 c_3 c_4 c_5] .$$

This is easily done using symbolic manipulation programs such as Maple/Mathematica.

C Evaluation of integrals

Here we provide the details of the computation of Feynman diagrams in Figure 2 and Figure 3. Figure 2a gives the following integral.

$$\begin{aligned} & \int d^2 \theta_1 d^4 \theta_2 d^4 \theta_3 \int \frac{d^D q}{(2\pi)^D} \frac{\Phi_I(-p_2 - p_3, \theta_1) \Phi_J(p_2, \theta_2) \Phi_K(p_3, \theta_3)}{q^2 (q - p_2)^2 (q - p_2 - p_3)^2} \\ & (-1) \bar{D}_1^2 D_2^2 [q] \delta^4(\theta_{12}) \bar{D}_1^2 D_3^2 [q + p_1] \delta^4(\theta_{13}) \delta^4(\theta_{32}) . \end{aligned} \tag{C.1}$$

We convert all the Grassmann integrations over $d^2\theta$ and $d^2\bar{\theta}$ into $d^4\theta$ by using up factors of \bar{D}^2 and D^2 respectively and integrate the δ -functions out.

$$\begin{aligned}
& \int d^4\theta_1 d^4\theta_2 \int \frac{d^D q}{(2\pi)^D} \frac{\Phi_I(-p_2 - p_3, \theta_1) D_2^2(\Phi_J(p_2, \theta_2) \Phi_K(p_3, \theta_2))}{q^2(q - p_2)^2(q - p_2 - p_3)^2} \\
& (-1) \delta^4(\theta_{12}) \bar{D}_1^2 D_2^2[q + p_1] \delta^4(\theta_{12}) \\
& = - \int d^2\theta \int \frac{d^D q}{(2\pi)^D} \frac{\Phi_I(-p_2 - p_3, \theta) \bar{D}^2 D^2(\Phi_J(p_2, \theta) \Phi_K(p_3, \theta))}{q^2(q - p_2)^2(q - p_2 - p_3)^2} \\
& = - \int d^2\theta \Phi_I(-p_2 - p_3, \theta) \Phi_J(p_2, \theta) \Phi_K(p_3, \theta) \\
& \quad \int \frac{d^D q}{(2\pi)^D} \frac{p_1^2}{q^2(q - p_2)^2(q - p_2 - p_3)^2} \tag{C.2}
\end{aligned}$$

We have simplified the expressions involving D -operator, using the identities given in appendix A. Figure 2b contributes the integral

$$\begin{aligned}
& \int d^2\theta_1 d^4\theta_2 d^4\theta_3 d^4\theta_4 d^4\theta_5 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(-p_2 - p_3, \theta_1) \Phi_J(p_2, \theta_2) \Phi_K(p_3, \theta_3)}{k^2 q^2 (k - q)^2 (q - p_2)^2 (q + p_1)^2 (k + p_1)^2} \\
& \bar{D}_1^2 D_4^2[k] \delta^4(\theta_{14}) \bar{D}_4^2 D_2^2[q] \delta^4(\theta_{24}) \bar{D}_5^2 D_3^2[q + p_1] \delta^4(\theta_{35}) \delta^4(\theta_{23}) \delta^4(\theta_{45}) \\
& \quad \bar{D}_1^2 D_5^2[k + p_1] \delta^4(\theta_{15}) \tag{C.3}
\end{aligned}$$

which is again simplified as above to obtain

$$\begin{aligned}
& \int d^2\theta \Phi_I(-p_2 - p_3, \theta) \Phi_J(p_2, \theta) \Phi_K(p_3, \theta) \\
& \int \frac{d^D q}{(2\pi)^D} \left[\frac{p_1^4}{k^2 q^2 (k - q)^2 (q - p_2)^2 (q + p_1)^2 (k + p_1)^2} - \frac{p_1^2}{k^2 q^2 (k - q)^2 (q - p_2)^2 (k - p_2)^2} \right] \tag{C.4}
\end{aligned}$$

The blob in Figure 2c consists of a pure chiral superfield loop as well as one involving gluons. The diagram with a blob (loop) made up of two gluons and one chiral propagators contributes

$$\begin{aligned}
& \int d^2\theta_1 d^4\theta_2 d^4\theta_3 d^4\theta_4 d^4\theta_5 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(-p_2 - p_3, \theta_1) \Phi_J(p_2, \theta_2) \Phi_K(p_3, \theta_3)}{k^2 q^2 (k - q)^2 (q - p_2)^2 (k - p_2)^2 (k + p_1)^2} \\
& (-1) \bar{D}_1^2 D_4^2[k] \delta^4(\theta_{14}) \bar{D}_1^2 D_3^2[k + p_1] \delta^4(\theta_{13}) \bar{D}_4^2 D_2^2[q] \delta^4(\theta_{24}) \delta^4(\theta_{25}) \delta^4(\theta_{35}) \delta^4(\theta_{45}) \tag{C.5}
\end{aligned}$$

which easily reduces to

$$- p_1^2 \int d^2\theta \int \frac{d^D q}{(2\pi)^D} \frac{(\Phi_I(-p_2 - p_3, \theta) \Phi_J(p_2, \theta) \Phi_K(p_3, \theta))}{k^2 q^2 (k - q)^2 (q - p_2)^2 (k - p_2)^2 (k + p_1)^2} \tag{C.6}$$

Contribution from Figure 2d is the integral

$$\int d^2\theta_1 d^4\theta_2 d^4\theta_3 d^4\theta_4 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(-p_2 - p_3, \theta_1) \Phi_J(p_2, \theta_2) \Phi_K(p_3, \theta_3)}{k^2 q^2 (k - q - p_2)^2 (q - p_3)^2 (k + p_1)^2} \\ \bar{D}_1^2 D_2^2[k] \delta^4(\theta_{12}) \bar{D}_1^2 D_4^2[k + p_1] \delta^4(\theta_{14}) \bar{D}_4^2 D_3^2[q - p_3] \delta^4(\theta_{34}) \delta^4(\theta_{24}) \delta^4(\theta_{23}) \quad (C.7)$$

which when simplified reduces to

$$\int d^2\theta \Phi_I(-p_2 - p_3, \theta) \Phi_J(p_2, \theta) \Phi_K(p_3, \theta) \int \frac{d^D q}{(2\pi)^D} \frac{p_3^2}{k^2 q^2 (k - q)^2 (q - p_3)^2 (k - p_3)^2} \quad (C.8)$$

The details of the evaluation of the integral from Figure 3 is given below as the exact value of this integral is very crucial.

$$\int d^4\theta_1 d^4\theta_2 d^4\theta_3 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(p_2 + p_3, \theta_1) \Phi_J(-p_2, \theta_2) \Phi_K(-p_3, \theta_3)}{k^2 q^2 (k - q)^2 (q - p_2)^2 (k - q - p_2)^2 (k - p_2 - p_3)^2} \\ \bar{D}_2^2 D_1^2[q] \delta^4(\theta_{12}) D_1^2[k - q] \delta^4(\theta_{13}) \bar{D}_1^2 D_2^2[k - p_1] \delta^4(\theta_{12}) \bar{D}_3^2 D_2^2[q] \delta^4(\theta_{32}) . \quad (C.9)$$

We can integrate the D -operators by parts and simplify this by getting rid of the Grassmann integrals one by one.

$$\int d^4\theta_1 d^4\theta_2 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{D_1^2 \Phi_I(p_2 + p_3, \theta_1) \Phi_J(-p_2, \theta_2) \Phi_K(-p_3, \theta_1)}{k^2 q^2 (k - q)^2 (q - p_2)^2 (k - q - p_2)^2 (k - p_2 - p_3)^2} \\ D_2^2 \bar{D}_2^2 \bar{D}_1^2 D_1^2[q] \delta^4(\theta_{12}) \delta^4(\theta_{12}) \bar{D}_1^2 D_2^2[k - p_1] \delta^4(\theta_{12}) \quad (C.10)$$

Using the identities in appendix A and rewriting the integral over $d^4\theta$ as a chiral integral

$$\int d^2\theta \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\bar{D}^2 D^2 \Phi_I(p_2 + p_3, \theta) \Phi_J(-p_2, \theta) \Phi_K(-p_3, \theta)}{k^2 q^2 (k - q)^2 (q - p_2)^2 (k - q - p_2)^2 (k - p_2 - p_3)^2}$$

Setting $p_2 = 0$ and re-labelling $r = k - q$ and using $\bar{D}^2 D^2(p) \Phi(p, \theta) = p^2 \Phi(p, \theta)$,

$$\int d^2\theta p_3^2 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{\Phi_I(p_3, \theta) \Phi_J(p_3, \theta) \Phi_K(-p_3, \theta)}{k^2 r^2 (k - r)^2 (r - p_3)^2 (k - p_3)^2} \\ = \mathcal{K} \int d^2\theta \Phi_I(p_3, \theta) \Phi_J(p_3, \theta) \Phi_K(-p_3, \theta) \quad (C.11)$$

where \mathcal{K} is the finite integral

$$\mathcal{K} \equiv p_3^2 \int \frac{d^D k d^D q}{(2\pi)^{2D}} \frac{1}{k^2 r^2 (k - r)^2 (r - p_3)^2 (k - p_3)^2} = \frac{\kappa}{(16\pi^2)^2} \quad (C.12)$$

with $\kappa = 6\zeta(3)$.

References

- [1] R. G. Leigh and M. J. Strassler, “Exactly Marginal Operators And Duality In Four-Dimensional $N=1$ Supersymmetric Gauge Theory,” Nucl. Phys. B **447** (1995) 95 [arXiv:hep-th/9503121].
- [2] O. Lunin and J. Maldacena, “Deforming field theories with $U(1) \times U(1)$ global symmetry and their gravity JHEP **0505**, 033 (2005) [arXiv:hep-th/0502086].
- [3] D. Z. Freedman and U. Gursoy, “Comments on the beta-deformed $N = 4$ SYM theory,” JHEP **0511** (2005) 042 [arXiv:hep-th/0506128].
- [4] S. Penati, A. Santambrogio and D. Zanon, “Two-point correlators in the beta-deformed $N = 4$ SYM at the next-to-leading order,” JHEP **0510** (2005) 023 [arXiv:hep-th/0506150].
- [5] A. Mauri, S. Penati, A. Santambrogio and D. Zanon, “Exact results in planar $N = 1$ superconformal Yang-Mills theory,” JHEP **0511** (2005) 024 [arXiv:hep-th/0507282].
- [6] A. Mauri, S. Penati, M. Pirrone, A. Santambrogio, D. Zanon, “On the perturbative chiral ring for marginally deformed $N = 4$ SYM theories,” JHEP **0608** (2006) 072 [arXiv:hep-th/0605145].
- [7] K. Madhu and S. Govindarajan, “Chiral primaries in the Leigh-Strassler deformed $N=4$ SYM – a perturbative study,” JHEP **0705** (2007) 038 [arXiv:hep-th/0703020].
- [8] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, “Beta Function In Supersymmetric Gauge Theories: Instantons Versus Traditional Approach,” Phys. Lett. B **166** (1986) 329 [Sov. J. Nucl. Phys. **43** (1986 YAFIA,43,459-464.1986) 294.1986 YAFIA,43,459].
- [9] O. Aharony, B. Kol and S. Yankielowicz, “On exactly marginal deformations of $N = 4$ SYM and type IIB supergravity on $AdS(5) \times S^{**5}$,” JHEP **0206** (2002) 039 [arXiv:hep-th/0205090].

- [10] M. Wijnholt, “Parameter space of quiver gauge theories,” arXiv:hep-th/0512122.
- [11] I. Jack, D. R. T. Jones and C. G. North, “ $N = 1$ supersymmetry and the three loop anomalous dimension for the chiral superfield,” Nucl. Phys. B **473** (1996) 308 [arXiv:hep-ph/9603386].
- [12] O. Aharony and S. S. Razamat, “Exactly marginal deformations of $N = 4$ SYM and of its supersymmetric orbifold descendants,” JHEP **0205** (2002) 029 [arXiv:hep-th/0204045].
- [13] N. Arkani-Hamed and H. Murayama, “Holomorphy, rescaling anomalies and exact beta functions in supersymmetric gauge theories,” JHEP **0006** (2000) 030 [arXiv:hep-th/9707133].
- [14] N. Arkani-Hamed and H. Murayama, “Renormalization group invariance of exact results in supersymmetric gauge theories,” Phys. Rev. D **57** (1998) 6638 [arXiv:hep-th/9705189].
- [15] A. Parkes and P. C. West, “Finiteness In Rigid Supersymmetric Theories,” Phys. Lett. B **138**, (1984) 99.
- [16] A. J. Parkes and P. C. West, “Three Loop Results In Two Loop Finite Supersymmetric Gauge Theories,” Nucl. Phys. B **256**, (1985) 340.
- [17] P. C. West, “The Yukawa Beta Function In $N=1$ Rigid Supersymmetric Theories,” Phys. Lett. B **137** (1984) 371.
- [18] D. R. T. Jones and L. Mezincescu, “The Beta Function In Supersymmetric Yang-Mills Theory,” Phys. Lett. B **136**, (1984) 242.
- [19] D. R. T. Jones and L. Mezincescu, “The Chiral Anomaly And A Class Of Two Loop Finite Supersymmetric Gauge Theories,” Phys. Lett. B **138**, (1984) 293.
- [20] A. J. Parkes, “Three Loop Finiteness Conditions In $N=1$ Superyang-Mills,” Phys. Lett. B **156**, (1985) 73.

- [21] I. Jack, D. R. T. Jones and P. C. West, “Not the no - renormalization theorem?,” Phys. Lett. B **258** (1991) 382.
- [22] A. Pickering and P. C. West, “The One Loop Effective Super-Potential and Non-Holomorphicity,” Phys. Lett. B **383**, 54 (1996) [arXiv:hep-th/9604147].
- [23] M. T. Grisaru, M. Rocek and R. von Unge, “Effective Kähler Potentials,” Phys. Lett. B **383** (1996) 415 [arXiv:hep-th/9605149].
- [24] R. Argurio, G. Ferretti and R. Heise, “An introduction to supersymmetric gauge theories and matrix models,” Int. J. Mod. Phys. A **19** (2004) 2015 [arXiv:hep-th/0311066].
- [25] R.C. Leng, “The McKay correspondence and orbifold Riemann-Roch,” PhD thesis (U. Warwick, 2002) available at the URL: <http://www.maths.warwick.ac.uk/%7Emiles/doctors/Leng/>
- [26] W.M. Fairbairn, T. Fulton and W.H. Klink, J. Math. Phys. **5** (1964) 1038; ibid. **23** (1982) 1747.
- [27] A. Bovier, M. Luling and D. Wyler, “Finite Subgroups Of SU(3),” J. Math. Phys. **22** (1981) 1543(Erratum-ibid. **23** (1982) 1992).